1 Functions of A Complex Variables I

Functions of a complex variable provide us some powerful and widely useful tools in in theoretical physics.

- Some important physical quantities are complex variables (the wave-function Ψ) $E_n \to E_n^0 + i\Gamma$
- Evaluating definite integrals.
- Obtaining asymptotic solutions of differentials equations.
- Integral transforms
- Many Physical quantities that were originally real become complex as simple theory is made more general. The energy ($1/\Gamma$ $\to$ the finite life time).

1.1 *Complex Algebra*

We here go through the complex algebra briefly.
A complex number z = (x,y) = x + iy, Where. $i = \sqrt{-1}$
We will see that the ordering of two real numbers (x,y) is significant, i.e. in general x + iy ≠ y + ix

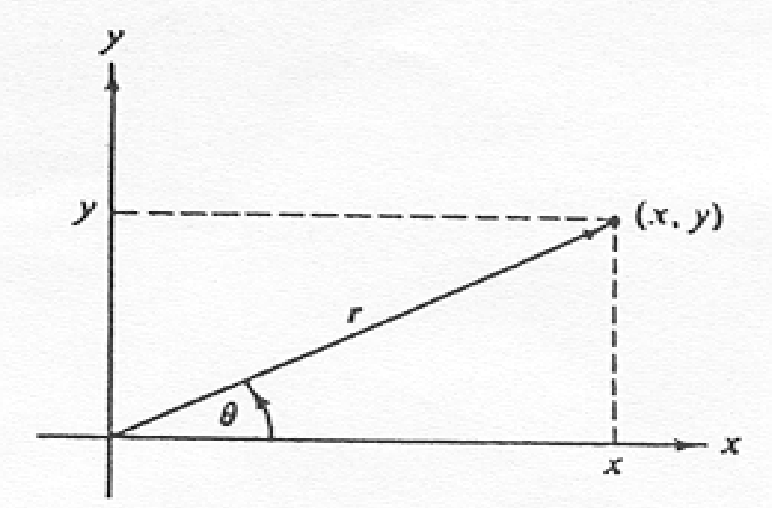
X: the real part, labeled by **Re(z); **y: the imaginary part, labeled by **Im(z)**

Three frequently used representations:

(1) Cartesian representation: x+iy

(2) polar representation, we may write
z=r(cos θ + i sinθ) or $z = r \cdot e^{i\theta}$

r – the modulus or magnitude of z
θ - the argument or phase of z



r – the modulus or magnitude of z

θ - the argument or phase of z

The relation between Cartesian and polar representation:

$$r = |z| = \left(x^2 + y^2\right)^{1/2}$$
$$\theta = \tan^{-1}(y/x)$$

The choice of polar representation or Cartesian representation is a matter of convenience. Addition and subtraction of complex variables are easier in the Cartesian representation. Multiplication, division, powers, roots are easier to handle in polar form,

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_2)$$
$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$
$$z_1 / z_2 = (r_1 / r_2) e^{i(\theta_1 - \theta_2)}$$
$$z^n = r^n e^{in\theta}$$

Using the polar form,

$$|z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

From z, complex functions f(z) may be constructed. They can be written

f(z) = u(x,y) + iv(x,y)

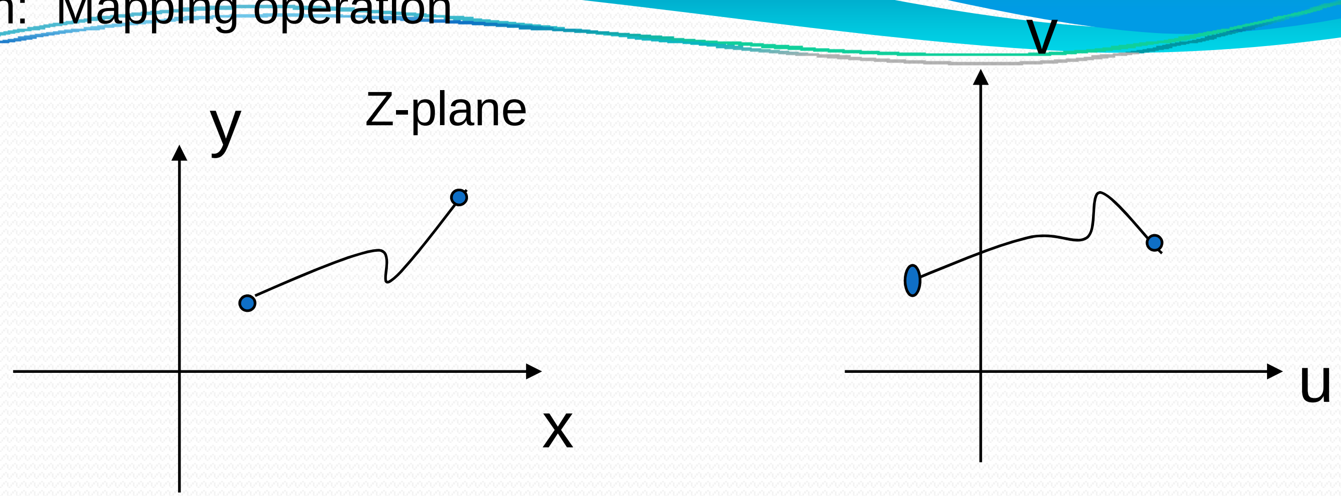
in which v and u are real functions.

For example if $f(z) = z^2$, we have

$$f(z) = (x^2 - y^2) + i2xy$$

The relationship between z and f(z) is best pictured as a mapping operation, we address it in detail later.

Function: Mapping operation



The function *w*(x,y)=*u*(x,y)+i*v*(x,y) maps points in the *xy* plane into points in the *uv* plane.

Since

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n$$

We get a not so obvious formula

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

Complex Conjugation: replacing i by –i, which is denoted by (*),

$$z^* = x - iy$$

We then have

$$zz^* = x^2 + y^2 = r^2$$

Hence

$$|z| = \left(zz^*\right)^{1/2}$$

Note:

Special features: single-valued function of a real variable ---- multi-valued function

$$z = re^{i\theta} \qquad re^{i(\theta+2n\pi)}$$

$$\ln z = \ln r + i\theta \qquad \ln z = \ln r + i(\theta + 2n\pi)$$

ln z is a multi-valued function. To avoid ambiguity, we usually set n=0 and limit the phase to an interval of length of 2π. The value of lnz with n=0 is called the principal value of lnz.

Another possibility

$|\sin x|, |\cos x| \leq 1$ for a real x;

however, possibly $|\sin z|, |\cos z| > 1$ and even $\to \infty$

Question:

Using the identities :

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}; \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

to show (a) $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$$

(b) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

1.2 Cauchy – Riemann conditions

Having established complex functions, we now proceed to differentiate them. The derivative of f(z), like that of a real function, is defined by

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z)$$

provided that the limit is independent of the particular approach to the point z. For real variable, we require that $\lim_{x \to x_o^+} f'(x) = \lim_{x \to x_o^-} f'(x) = f'(x_o)$

,

Now, with z (or z_o) some point in a plane, our requirement that the limit be independent of the direction of approach is very restrictive.

Consider

$$\delta z = \delta x + i\delta y$$

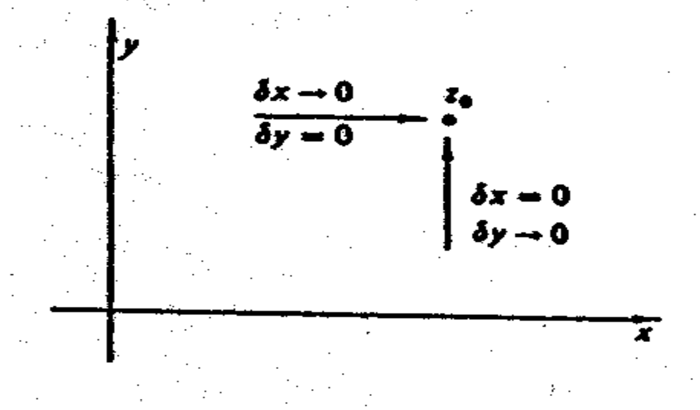
$$\delta f = \delta u + i\delta v$$

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

Let us take limit by the two different approaches as in the figure. First, with δy = 0, we let δx→0,

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$



Assuming the partial derivatives exist. For a second approach, we set δx = 0 and then let δy→ 0. This leads to

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

If we have a derivative, the above two results must be identical. So,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are the famous Cauchy-Riemann conditions. These Cauchy-Riemann conditions are necessary for the existence of a derivative, that is, if $f'(x)$ exists, the C-R conditions must hold.

Conversely, if the C-R conditions are satisfied and the partial derivatives of u(x,y) and v(x,y) are continuous, $f'(z)$ exists. (see the proof in the text book).

Analytic functions

If f(z) is differentiable at $z = z_0$ and in some small region around z_0 , we say that f(z) is analytic at $z = z_0$

Differentiable: Cauthy-Riemann conditions are satisfied
the partial derivatives of u and v are continuous

Analytic function:

Property 1: $$\nabla^2 u = \nabla^2 v = 0$$

Property 2: established a relation between u and v

Example:

$$\text{Find the analytic functions } w(z) = u(x, y) + iv(x, y)$$

$$\text{if } (a)\ u(x, y) = x^3 - 3xy^2$$

$$(b)\ v(x, y) = e^{-y} \sin x$$

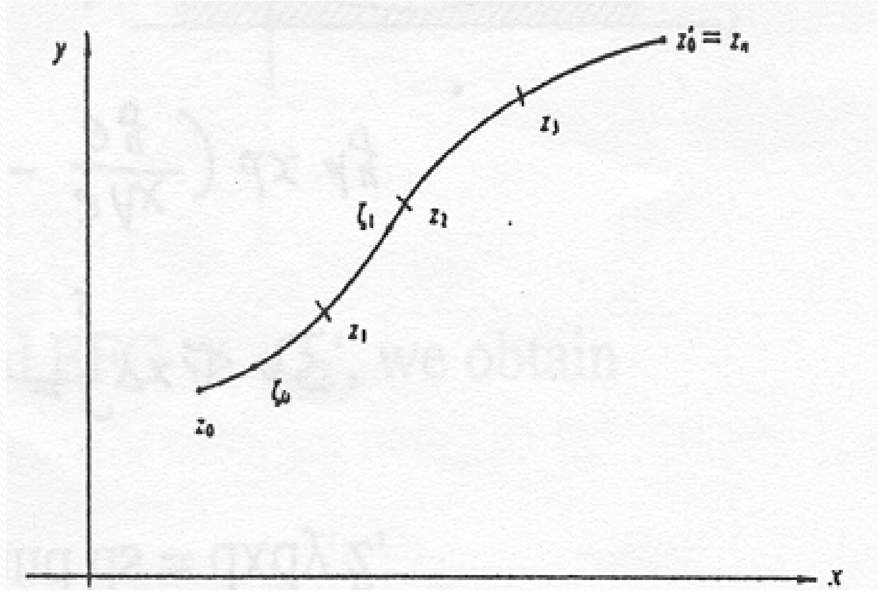
1.3 Cauchy’s integral Theorem

We now turn to integration.

in close analogy to the integral of a real function

The contour $z_0 \to z_0'$ is divided into n intervals .Let $n \to \infty$ wiith $|\Delta z_j| = |z_j - z_{j-1}| \to 0$ for j. Then

$$\lim_{n\to\infty}\sum_{j=1}^{n} f(\zeta_j)\Delta z_j = \int_{z_0}^{z_0'} f(z)dz$$



provided that the limit exists and is independent of the details of choosing the points z_j and ζ_j, where ζ_j is a point on the curve bewteen z_j and z_{j-1}.

The right-hand side of the above equation is called the contour (path) integral of f(z)

As an alternative, the contour may be defined by

$$\int_{c\ z_1}^{z_2} f(z)dz = \int_{c\ x_1y_1}^{x_2y_2} [u(x,y)+iv(x,y)][dx+idy]$$

$$= \int_{c\ x_1y_1}^{x_2y_2} [udx - vdy] + i \int_{c\ x_1y_1}^{x_2y_2} [vdx + udy]$$

with the path C specified. This reduces the complex integral to the complex sum of real integrals. It's somewhat analogous to the case of the vector integral.

An important example $\int_c z^n dz$

where C is a circle of radius r>0 around the origin z=0 in the direction of counterclockwise.

In polar coordinates, we parameterize $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$, and have

$$\frac{1}{2\pi i}\int_{c} z^{n}\,dz = \frac{r^{n+1}}{2\pi}\int_{0}^{2\pi}\exp\left[i(n+1)\theta\right]d\theta$$

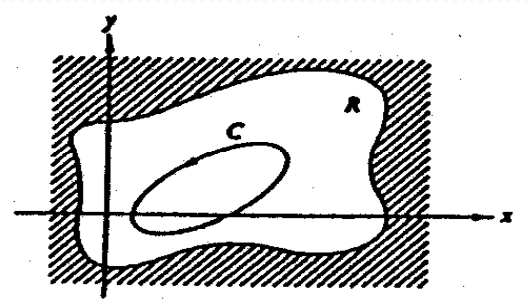
$$= \begin{cases} 0 & \text{for } n \neq -1 \\ 1 & \text{for } n = -1 \end{cases}$$

which is independent of r.

Cauchy's integral theorem

- If a function f(z) is analytical (therefore single-valued) [and its partial derivatives are continuous] through some simply connected region **R**, for every closed path C in **R**,

$$\oint_{c} f(z)\,dz = 0$$



Stokes's theorem proof

Proof: (under relatively restrictive condition: the partial derivative of u, v are continuous, which are actually not required but usually satisfied in physical problems)

$$\oint_c f(z)dz = \oint_c (udx - vdy) + i\oint_c (vdx + udy)$$

These two line integrals can be converted to surface integrals by Stokes's theorem

$$\oint_c \underline{A} \cdot d\,\underline{l} = \int_s \nabla \times \underline{A} \cdot d\,\underline{s}$$

Using $\underline{A} = A_x \hat{x} + A_y \hat{y}$ and $ds = dxdy\ \hat{z}$

We have

$$\oint_c (A_x dx + A_y dy) = \oint_c \underline{A} \cdot d\underline{l} = \int_s \nabla \times \underline{A} \cdot d\underline{s}$$

$$= \int \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dxdy$$

For the real part, If we let u = A_x, and v = -A_y, then

$$\oint_c (udx - vdy) = -\int\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdy$$

=0 [since C-R conditions $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$]

For the imaginary part, setting u = A_y and v = A_x, we have

$$\oint (vdx + udy) = \int\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy = 0$$

$$\oint f(z)dz = 0$$

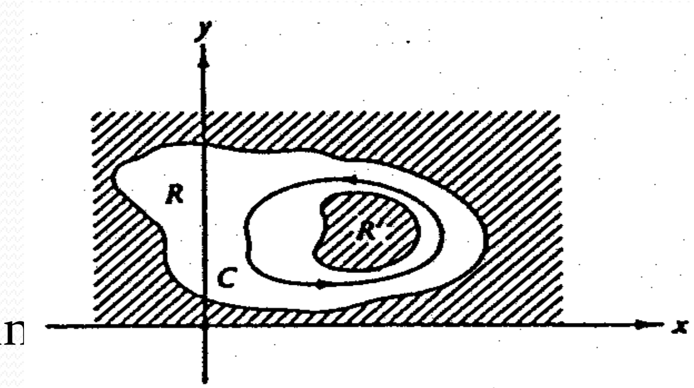
As for a proof without using the continuity condition, see the text book.

The consequence of the theorem is that for analytic functions the line integral is a function only of its end points, independent of the path of integration,

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)dz$$

•*Multiply connected regions*

The original statement of our theorem demanded a simply connected region. This restriction may easily be relaxed by the creation of a barrier, a contour line. Consider the multiply connected region of Fig.1.6 In which f(z) is not defined for the interior R′



1.6 Fig.

Cauchy's in … he contour C, but we can construct a C′ for which the theorem holds. If line segments DE and GA arbitrarily close together, then

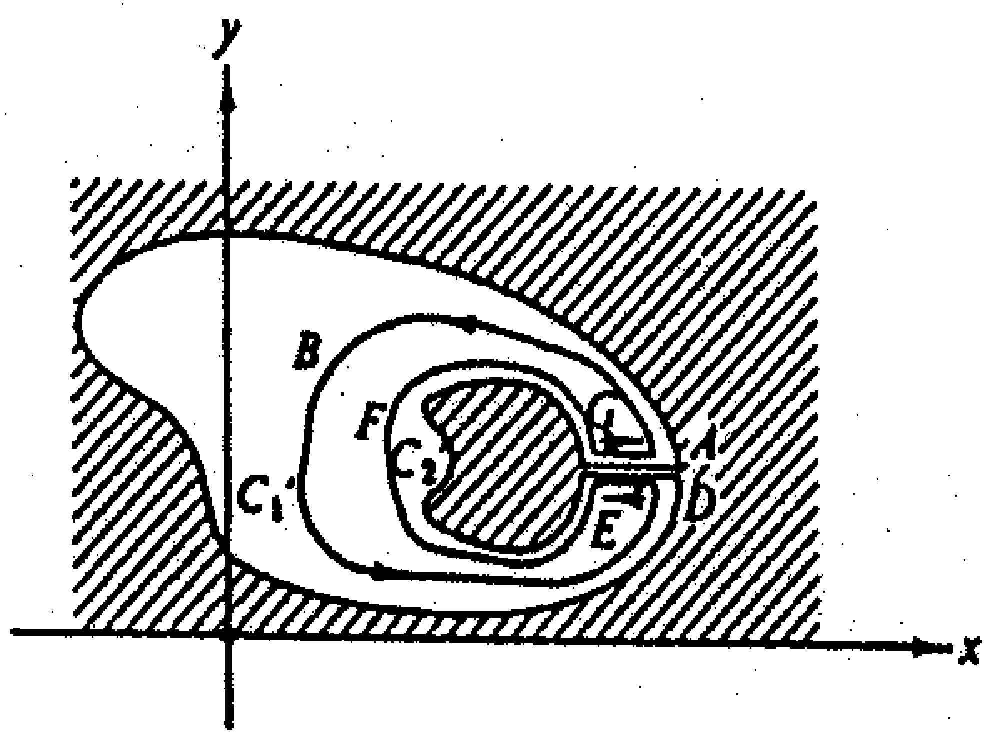
$$\int_G^A f(z)dz = -\int_D^E f(z)dz$$

$$\oint_{\substack{C' \\ (ABDEFGA)}} f(z)dz = \left[\int_{ABD} + \int_{DE} + \int_{GA} + \int_{EFG} \right] f(z)dz$$

$$= \left[\int_{\mathrm{ABD}} + \int_{\mathrm{EFG}} \right] \mathrm{f}(\mathrm{z})\mathrm{dz} = 0$$

$$\oint_{C_1'} f(z)dz = \oint_{C_2'} f(z)dz$$

$$ABD \to C_1' \quad EFG \to -C_2'$$



1.4 Cauchy's Integral Formula

Cauchy's integral formula: If f(z) is analytic on and within a closed contour C then

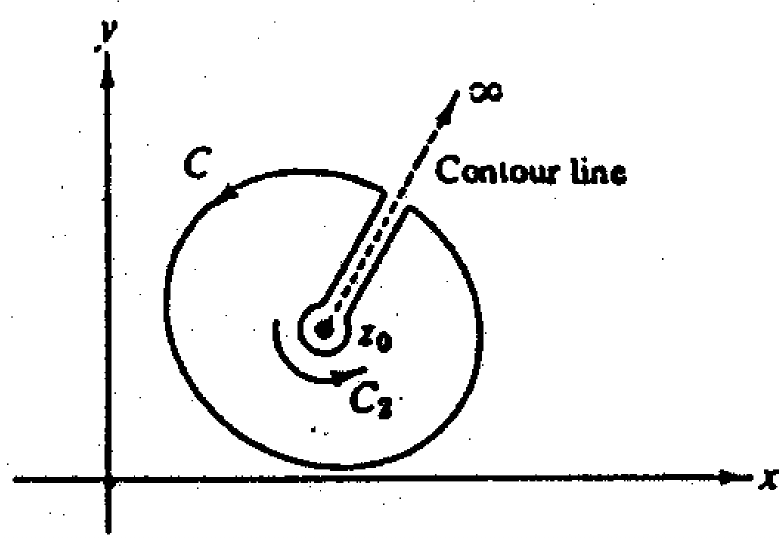
$$\oint_C \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0)$$

in which z_0 is some point in the interior region bounded by C. Note that here $z-z_0 \neq 0$ and the integral is well defined.

Although f(z) is assumed analytic, the integrand $(f(z)/z-z_0)$ is not analytic at $z=z_0$ unless $f(z_0)=0$. If the contour is deformed as in Fig.1.8 Cauchy's integral theorem applies.

So we have

$$\oint_C \frac{f(z)dz}{z-z_0} - \oint_{C_2} \frac{f(z)}{z-z_0}dz = 0$$



Let $z-z_0=re^{i\theta}$, here r is small and will eventually be made to approach zero

$$\oint_{C_2}\frac{f(z)dz}{z-z_0}dz=\oint_{C_2}\frac{f\left(z_0+re^{i\theta}\right)}{re^{i\theta}}rie^{i\theta}d\theta$$

$$(r\to o)\qquad =if(z_0)\oint_{C_2}d\theta=2\pi if(z_0)$$

Here is a remarkable result. The value of an analytic function is given at an interior point at $z=z_o$ once the values on the boundary C are specified.

What happens if z_o is exterior to C?

In this case the entire integral is analytic on and within C, so the integral vanishes.

$$\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-z_0} = \begin{cases} f(z_0), & z_0 \text{ interior} \\ 0, & z_0 \text{ exterior} \end{cases}$$

Derivatives

Cauchy's integral formula may be used to obtain an expression for the derivation of f(z)

$$f'(z_0) = \frac{d}{dz_0}\left(\frac{1}{2\pi i}\oint \frac{f(z)\,dz}{z-z_0}\right)$$

$$= \frac{1}{2\pi i}\oint f(z)dz \frac{d}{dz_0}\left(\frac{1}{z-z_0}\right) = \frac{1}{2\pi i}\oint \frac{f(z)dz}{(z-z_0)^2}$$

Moreover, for the n-th order of derivative

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)dz}{(z-z_0)^{n+1}}$$

We now see that, the requirement that f(z) be analytic not only guarantees a first derivative but derivatives of all orders as well! The derivatives of f(z) are automatically analytic. Here, it is worth to indicate that the converse of Cauchy's integral theorem holds as well

Morera's theorem:

If a function f(z) is continuous in a simply connected region R and $\oint_C f(z)dz = 0$ for every closed C within R, then f(z) is analytic throught R (see the text book).

$\therefore$

Examples

1. If $f(z) = \sum_{n\geq 0} \mathrm{a}_{\mathrm{n}} z^n$ is analytic on and within a circle about the origin, find a_n.

$$f^{(j)}(z) = j!\,a_j + \sum_{n-j\geq 1} a_n \{\ \} z^{n-j}$$

$$f^{(j)}(0) = j!\,a_j$$

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint \frac{f(z)dz}{z^{n+1}}$$

2.In the above case, $|f(z)| \le M$ on a circle of radius r about the origin,

then $|a_n| r^n \le M$ (Cauchy's inequality)

Proof:

$$|a_n| = \frac{1}{2\pi}\left|\oint_{|z|=r} \frac{f(z)dz}{z^{n+1}}\right| \le M(r)\frac{2\pi r}{2\pi r^{n+1}} \le \frac{M}{r^n}$$

where $M(r) = Max_{|z|=r} |f(r)|$

3. Liouville's theorem: If f(z) is analytic and bounded in the complex plane, it is a constant.

Proof: For any zo, construct a circle of radius R around z_0,

$$|f'(z_0)| = \left|\frac{1}{2\pi i}\oint_R \frac{f(z)dz}{(z - z_0)^2}\right| \le \frac{M}{2\pi}\frac{2\pi R}{R^2}$$

$$= \frac{M}{R}$$

Since R is arbitrary, let $R \to \infty$, we have

$$f'(z) = 0, \text{ i.e, } f(z) = \text{const} .$$

Conversely, the slightest deviation of an analytic function from a constant value implies that there must be at least one singularity somewhere in the infinite complex plane. Apart from the trivial constant functions, then, singularities are a fact of life, and we must learn to live with them, and to use them further.

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

Here C may be any contour with the annular region $r < |z-z_0| < R$ encircling z_0 once in a counterclockwise sense.

Laurent Series need not to come from evaluation of contour integrals. Other techniques such as ordinary series expansion may provide the coefficients.

Numerous examples of Laurent series appear in the next chapter.

Assignment 1

- Try yourself

Q.1 : Find real & Imaginary parts of (i) e^{z^2} **(ii)** e^{e^z}

Q.2 : If α and β are the imaginary cube roots of unity, prove that $\alpha e^{\alpha x} + \beta e^{\beta x} = -e^{\frac{-x}{2}}\left(\cos\frac{\sqrt{3}}{2}x + \sqrt{3}\sin\frac{\sqrt{3}}{2}x\right)$

Q.3 : Show that $\log_e \frac{3-i}{3+i} = 2i\left(n\pi - \tan^{-1}\frac{1}{3}\right)$

Q.4: Show that log(6+8i)= log 10 + i $\tan^{-1}\frac{4}{3}$

Q.5 :Prove that $\sqrt{i}^{\sqrt{i}} = e^{\frac{-\pi}{4\sqrt{2}}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$

Assignment 2

- Try yourself

Q.1 : if u=$\log\tan(\frac{\pi}{4}+\frac{\theta}{2})$, then prove that

(i) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

(ii) $\cosh u = \sec\theta$

Q.2: Prove that (i) $\overline{\sin z} = \sin\bar{z}$ (ii) $\overline{\tan z} = \tan\bar{z}$

(iii) $\overline{\cos z} = \cos\bar{z}$

Q.3 : If $\tan(\theta + i\varphi) = \tan\alpha + i\sec\alpha$ show

$e^{2\varphi} = \mp\cot\frac{\alpha}{2}$ and $2\theta = \left(n+\frac{1}{2}\right)\pi + \alpha$

Q.4: find the value of f(i) so that the function

f(z)= $\frac{iz^3-1}{z-i}$ $is\ not\ continuous\ at\ z = i$

Q.5: If $f(z) = \frac{yx^3(y-ix)}{y^2+x^6}$ $, z \neq 0\ \&\ f(0) = 0, prove\ that$

$\frac{f(z)-f(0)}{z} \rightarrow 0$ as z$\rightarrow 0$ along any radius vector but not as z $\rightarrow 0$

In any manner.

Assignment 3

- Try yourself

Q.1:Determine the Analytic function whose real part is

(i) $e^x[(x^2 - y^2)\cos y - 2xy\sin y]$

(ii) $\log\sqrt{(x^2 + y^2)}$

Q.2:Determine the Regular function whose imaginary part is

(i) $e^{-x}(x\cos y + y\sin y)$

(ii) $\cos y \sinh x$

Q.3: if $f(z) = u + iv$ **is an analytic function ,find f(z) if**

$u - v = e^x(\cos y - \sin y)$

Q.4: Show that the function $v(x,y) = \ln(x^2 + y^2) + x - 2y$

Is harmonic . find its conjugate harmonic function u(x,y) and the corresponding analytic function f(z)

Q.5 If f(z) is an analytic function of z , prove that

(i) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\, f(z)|^2 = 2|f'(z)|^2$

(ii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Assignment 4

- Try yourself

Q.1 Evaluate $\int_0^{1+i}(x^2 - iy)$ **dz along the paths y = x**

Q.2 Show that ***Evaluate the integral*** $\oint_c \frac{(\cos \pi z^2 + \sin \pi z^2)dz}{(z-2)(z-1)^2}$ **c:** $|z| = 3$ **by Cauchy's integral formula.**

Q.3 ***Evaluate the integral*** $\oint_c \frac{(1-2z)\,dz}{z(z-2)(z-1)}$ **c:** $|z| = \frac{1}{2}$ **by Cauchy 's integral formula.**

Q.4 $\oint_c \frac{z\,dz}{(z-1)(z-2)^2}$ **c:** $|z-2| = \frac{1}{2}$ **by Cauchy's integral formula.**

- **Thank you**